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Interference in Phase Space of Squeezed States for the Time-Dependent Hamiltonian System

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We showed that the idea of Schleich and Wheeler (1987, *Nature* **326**, 574) for the semiclassical approach of the interference in phase space of harmonic oscillator squeezed states can be extended to that of general time-dependent Hamiltonian system. The quantum phase properties of squeezed states for the general time-dependent Hamiltonian system are investigated by using the quantum distribution function. The weighted overlaps A_n and phases θ_n for the system are evaluated in the semiclassical limit.

KEY WORDS: interference in phase space; quantum distribution function; time-dependent Hamiltonian system; squeezed state.

1. INTRODUCTION

Schleich and Wheeler devised a technique of evaluating semiclassical overlap integral in the phase space interference of quantum states (Schleich *et al.*, 1988). The technique can not only be extended to the spherical phase space parameterized by the components of angular momentum (Lassig and Milburn, 1993) but can also be applied to the investigation of various nonclassical properties of quantum states such as photon distribution of squeezed states (Schleich *et al.*, 1988; Schleich and Wheeler, 1987a,b), interference in parabolic space (Chaturvedi *et al.*, 1998), and phase properties of Jaynes–Cumming model (El-Orany *et al.*, 2004). It has been realized that the interference in phase space of harmonic oscillator gives rise to various nonclassical effects which are fundamental features of quantum mechanics. In particular, the photon number probability distribution of squeezed state exhibits nonclassical oscillations with appropriate choice of the parameters, which can be well understood in terms of interference in phase space (Schleich *et al.*, 1988; Schleich and Wheeler, 1987a,b). The calculation of the overlap between bands of two quantum states is a central problem in quantum mechanics and gives insight in analyzing quantum probability amplitude. Krähmer *et al.*

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studied giant oscillations in the photon distribution of a highly squeezed state rotated relative to the momentum axis (Krähmer *et al.*, 1994). Chaturvedi *et al.* showed that the interference in the phase space algorithm can be extended to the hyperbolic space underlying the group of SU(1,1) Lie algebra which attracted great interest in many branches of physics (Chaturvedi *et al.*, 1998).

The main goal of the present paper is to investigate quantum phase properties of squeezed states for the time-dependent Hamiltonian system (TDHS) by finding the quantum distribution using the method employed in Schleich and Wheeler (1987a). The system whose Hamiltonian explicitly depends on time has attracted considerable attention (Bandyopadhyay *et al.*, 2001; Choi, 2004a; Dodonov and Man'ko, 1978; Kim and Lee, 2000; Nieto and Truax, 2000; Pedrosa and Guedes, 2002; Šamaj, 2002; Song, 2000; Um *et al.*, 2002; Wei *et al.*, 2002) for several decades because of its applications to various branches in physics. The dynamical invariant operator method has been widely employed to find exact quantum states of TDHS after the report of simple relation between the solutions of the Schrödinger equation and the eigenstates of dynamical invariants for the time-dependent harmonic oscillator by Lewis and Riesenfeld (1969). One of the typical types of the TDHS is Caldirola–Kanai oscillator (Kanai, 1948) which gives dissipative classical equation of motion.

This paper is organized as the follows. In Section 2, we study the quantum description of the TDHS using Lewis-Riesenfeld (LR) invariant operator. In Section 3, the squeezed state is investigated by solving the eigenvalue equation of $\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger$, where $\hat{a}$ is annihilation operator. The interference properties in the phase space of the squeezed state are described in Sections 4 and 5. We summarize the results of the paper in the last section.

2. TIME-DEPENDENT HAMILTONIAN SYSTEM

In this section, we study the invariant operator method employed in Choi (2004a) in order to derive the exact solution of the Schrödinger equation for the TDHS. The Hamiltonian describing general TDHS is

$$\hat{H}(\hat{q}, \hat{p}, t) = A(t)\hat{p}^2 + B(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + C(t)\hat{q}^2 + D(t)\hat{q} + E(t)\hat{p} + F(t), \tag{1}$$

where $A(t)$–$F(t)$ are some time-dependent functions. The introduction of the LR (Lewis–Riesenfeld) invariant operator (Lewis and Riesenfeld, 1969) may relieve the mathematical difficulties to evaluate exact quantum states of the system. From $d\hat{I}/dt = 0$, the LR invariant operator $\hat{I}$ can be represented as (Choi, 2004a)

$$\hat{I} = \frac{\Omega^2}{4\rho^2(t)}(\hat{q} - q_p(t))^2 + \left[\rho(\hat{p} - p_p(t)) + \frac{1}{2A}(2B\rho(t) - \dot{\rho}(t))(\hat{q} - q_p(t))\right]^2, \tag{2}$$

where Ω is real positive constant, $\rho(t)$ is some time-dependent classical solution of the following differential equation:

$$\ddot{\rho}(t) - \frac{\dot{A}}{A}\dot{\rho}(t) + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)\rho(t) - \Omega^2 A^2 \frac{1}{\rho^3(t)} = 0, \quad (3)$$

and $q_p(t)$ and $p_p(t)$ are particular solutions of the classical equation of motion in position and momentum space, respectively. Taking advantage of Hamiltonian dynamics we can readily show that $q_p(t)$ and $p_p(t)$ satisfies

$$\ddot{q}_p(t) - \frac{\dot{A}}{A}\dot{q}_p(t) + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)q_p(t)$$
$$= -\frac{\dot{A}E}{A} + 2BE - 2AD + \dot{E}, \quad (4)$$
$$\ddot{p}_p(t) - \frac{\dot{C}}{C}\dot{p}_p(t) + \left(4AC - 2\frac{\dot{C}B}{C} - 4B^2 + 2\dot{B}\right)p_p(t)$$
$$= \frac{\dot{C}D}{C} + 2BD - 2CE - \dot{D}. \quad (5)$$

In terms of an annihilation operator that defined as

$$\hat{a} = \sqrt{\frac{1}{\hbar\Omega}}\left\{\left[\frac{\Omega}{2\rho} + i\frac{1}{2A}(2B\rho - \dot{\rho})\right](\hat{q} - q_p) + i\rho(\hat{p} - p_p)\right\}, \quad (6)$$

and its conjugate operator $\hat{a}^\dagger$ which is a creation operator, Eq. (2) can be expressed in a simple form

$$\hat{I} = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \quad (7)$$

Note that $\hat{a}$ and $\hat{a}^\dagger$ satisfy boson commutation relation: $[\hat{a}, \hat{a}^\dagger] = 1$. Let's represent the eigenvalue equation of $\hat{I}$ as

$$\hat{I}|\phi_n(t)\rangle = \lambda_n|\phi_n(t)\rangle. \quad (8)$$

Then, the eigenvalues λ_n and position eigenstates $\langle q|\phi_n(t)\rangle$ are given by (Choi, 2004a):

$$\lambda_n = \hbar\Omega\left(n + \frac{1}{2}\right), \quad (9)$$

$$\langle q|\phi_n(t)\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{\Omega}{2\rho^2\hbar}}(q - q_p)\right]$$
$$\times \exp\left\{\frac{i}{\hbar}p_p q - \frac{1}{2\rho\hbar}\left[\frac{\Omega}{2}\frac{1}{\rho} + \frac{i}{2A}(2B\rho - \dot{\rho})\right](q - q_p)^2\right\}. \quad (10)$$

The wave functions $\langle q|\psi_n(t)\rangle$ that satisfy the Schrödinger equation are same as the eigenstates $\langle q|\phi_n(t)\rangle$ except for some time-dependent phase factors $\exp[i\epsilon_n(t)]$ (Lewis and Riesenfeld, 1969):

$$\langle q|\psi_n(t)\rangle = \langle q|\phi_n(t)\rangle \exp[i\epsilon_n(t)]. \tag{11}$$

In the above equation, the phases $\epsilon_n(t)$ are given by (Choi, 2004a):

$$\epsilon_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{A(t')\Omega}{\rho^2(t')}dt' - \frac{1}{\hbar}\int_0^t \left[\mathcal{L}_p(q_p(t'),\dot{q}_p(t'),t') - \frac{E^2(t')}{4A(t')} + F(t')\right]dt', \tag{12}$$

where

$$\mathcal{L}_p(q_p(t'),\dot{q}_p(t'),t') = \frac{1}{4A(t')}\dot{q}_p^2(t') - \frac{B(t')}{A(t')}q_p(t')\dot{q}_p(t') - \left(C(t') - \frac{B^2(t')}{A(t')}\right)q_p^2(t'). \tag{13}$$

The general solution of the Schrödinger equation may be expanded in terms of the Fock state wave functions

$$\langle q|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n\langle q|\psi_n(t)\rangle, \tag{14}$$

where c_n are the amplitudes of the nth wave functions that contribute to the wave packet. Taking advantage of the orthonormality of the $\langle q|\psi_n(t)\rangle$, we see that c_n can be evaluated from

$$c_n = \int_{-\infty}^{\infty}\langle\psi_n(t)|q\rangle\langle q|\psi(t)\rangle\,dq. \tag{15}$$

The probabilities for finding the energy of the system in nth quantum states are

$$P_n = |c_n|^2. \tag{16}$$

In quantum optics, P_n are often called photon distributions.

3. SQUEEZED STATE

Since the main purpose of this paper is to find the distribution of Eq. (16) in squeezed state for the general TDHS we derive exact squeezed state in this section. It is well known that the standard definition of coherent state $|\alpha\rangle$ is the eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{17}$$

On the other hand, the squeezed state $|\beta\rangle$ is the eigenstate of the operator $\hat{b}$ that defined by

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger, \tag{18}$$

where complex numbers μ and ν satisfy

$$|\mu|^2 - |\nu|^2 = 1. \tag{19}$$

To simplify the problem, we only consider for the case that μ and ν are real values. We can easily check that $\hat{b}$ satisfies $[\hat{b}, \hat{b}^\dagger] = 1$. The eigenvalue equation for $\hat{b}$ is

$$\hat{b}|\beta\rangle = \beta|\beta\rangle. \tag{20}$$

By multiplying both sides from the left by $\langle q|$, we easily derive the coordinate squeezed state

$$\langle q|\beta\rangle = N_q \exp\left\{ -\frac{1}{\rho\hbar}\left\{\left[\frac{\Omega}{2\rho}s + \frac{i}{2A}(2B\rho - \dot{\rho})\right]\left(\frac{1}{2}q^2 - q_p q\right) - i\rho p_p q\right\} + \frac{(s+1)\alpha + (s-1)\alpha^*}{2\rho}\sqrt{\frac{\Omega}{\hbar}}q\right\}, \tag{21}$$

where squeezing parameter s is defined by

$$s = \frac{\mu + \nu}{\mu - \nu}, \tag{22}$$

and N_q is the normalization constant given by

$$N_q = \left(\frac{\Omega s}{2\rho^2\hbar\pi}\right)^{1/4} \exp\left\{-\frac{\Omega s}{4\rho^2\hbar}\left(q_p + 2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)^2 + i\delta_{s,q}\right\}. \tag{23}$$

In the expression of Eq. (21), we used

$$\beta = \mu\alpha + \nu\alpha^*. \tag{24}$$

Since the absolute square of Eq. (21) is Gaussian

$$|\langle q|\beta\rangle|^2 = |N_q|^2 \exp\left\{-\frac{s\Omega}{2\rho^2\hbar}\left[q^2 - \left(2q_p + 4\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)q\right]\right\}, \tag{25}$$

we easily identify the variation of q as

$$\Delta q = \sqrt{\frac{\rho^2\hbar}{\Omega s}}. \tag{26}$$

Now we multiply both sides of Eq. (20) by $\langle p|$ in order to obtain the p space squeezed state:

$$\langle p|\beta\rangle = N_p \exp\left\{\frac{q_p}{i\hbar}p - \frac{1}{2\hbar}\left[\frac{\Omega}{2\rho}s + \frac{i}{2A}(2B\rho - \dot{\rho})\right]^{-1} \times \left[\rho(p^2 - 2p_p p) + i[(s+1)\alpha + (s-1)\alpha^*]\sqrt{\hbar\Omega}p\right]\right\}, \tag{27}$$

where normalization constant N_p is

$$N_p = \left(\frac{2\Omega}{\hbar\pi}\right)^{1/4}\left(\frac{s\Omega^2}{\rho^2} + \frac{(2B\rho - \dot{\rho})^2}{sA^2}\right)^{-1/4} \times \exp\left\{-\frac{\Omega}{\hbar}\left(\frac{s\Omega^2}{\rho^2} + \frac{(2B\rho - \dot{\rho})^2}{sA^2}\right)^{-1} \times \left[p_p + \frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha - \frac{\rho}{A\Omega}(2B\rho - \dot{\rho})\mathrm{Re}\,\alpha\right)\right]^2 + i\delta_{s,p}\right\}. \tag{28}$$

Then, we can see that the absolute square of Eq. (27) is also Gaussian

$$|\langle p|\beta\rangle|^2 = |N_p|^2 \exp\left\{-\frac{2\Omega}{\hbar}\left[\frac{s\Omega^2}{\rho^2} + \frac{1}{sA^2}(2B\rho - \dot{\rho})^2\right]^{-1} \times \left\{p^2 - 2\left[p_p + \frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha - \frac{\rho}{A\Omega}(2B\rho - \dot{\rho})\mathrm{Re}\,\alpha\right)\right]p\right\}\right\}, \tag{29}$$

so that

$$\Delta p = \left\{\frac{\hbar}{4\Omega}\left[\frac{s\Omega^2}{\rho^2} + \frac{1}{sA^2}(2B\rho - \dot{\rho})^2\right]\right\}^{1/2}. \tag{30}$$

Equations (26) and (30) mean the uncertainty relation is

$$\Delta q\,\Delta p = \frac{\hbar}{2}\left[1 + \frac{\rho^2}{s^2A^2\Omega^2}(2B\rho - \dot{\rho})^2\right]^{1/2}. \tag{31}$$

Note that the above equation is explicitly dependent on parameter s. The dependency of the squeezed state on the squeezing parameter for the TDHS is described in Choi (2004b). The simplest phase space distribution of squeezed state is

$$\begin{aligned} P_{\mathrm{s}}(q,p,t) &= |\langle q|\beta\rangle|^2 |\langle p|\beta\rangle|^2 \\ &= \frac{\Omega}{\hbar\pi\rho}\left(\frac{\Omega^2}{\rho^2}+\frac{(2B\rho-\dot{\rho})^2}{s^2A^2}\right)^{-1/2} \\ &\quad \times \exp\left\{-\frac{\Omega s}{2\rho^2\hbar}\left(q-q_p-2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)^2\right. \\ &\quad -\frac{2\Omega}{\hbar}\left(\frac{s\Omega^2}{\rho^2}+\frac{(2B\rho-\dot{\rho})^2}{sA^2}\right)^{-1} \\ &\quad \left.\times\left[p-p_p-\frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha-\frac{\rho}{A\Omega}(2B\rho-\dot{\rho})\mathrm{Re}\,\alpha\right)\right]^2\right\}. \end{aligned} \tag{32}$$

We will use the above equation in the derivation of semiclassical distribution function later.

4. QUANTUM DISTRIBUTION IN SQUEEZED STATE

The scalar product between Fock state $|\psi_n\rangle$ and squeezed state $|\beta\rangle$ determine the area of overlap in phase space. Thus, the probabilities that the system is in its nth energy eigenstate are

$$P_n = |\langle\psi_n|\beta\rangle|^2 = \left|\int_{-\infty}^{\infty}\langle\psi_n|q\rangle\langle q|\beta\rangle\right|^2. \tag{33}$$

After substituting Eqs. (11) and (21) into the above equation, the integration in the above equation can be performed with the help of the generating function

$$e^{-t^2+2xt} = \sum_n \frac{H_n(x)}{n!}t^n. \tag{34}$$

Thus, we finally arrive at

$$\begin{aligned} P_n &= \frac{2s^{1/2}}{1+s}\frac{1}{2^n n!}\left(\frac{s-1}{s+1}\right)^n H_n(y)H_n(y^*) \\ &\quad \times \exp\left\{-2s(\mathrm{Re}\,\alpha)^2+\frac{1+s^2}{2(1+s)}(\alpha^2+\alpha^{*2})+(s-1)|\alpha|^2\right\}, \end{aligned} \tag{35}$$

where

$$y = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{s+1}{s-1}}\alpha+\sqrt{\frac{s-1}{s+1}}\alpha^*\right). \tag{36}$$

In the calculation of Eq. (35), we can see that Eq. (35) is exactly the same as Eq. (3.3) of Schleich and Wheeler (1987a) by introducing $c = (s+1)/(s-1)$. Schleich and Wheeler (1987a) illustrate that this distribution function suffers oscillation for highly squeezed state. If α is real, y becomes

$$y = \frac{\sqrt{2}\alpha s}{\sqrt{s^2 - 1}}, \tag{37}$$

so that

$$P_n = \frac{2s^{1/2}}{1+s}\frac{1}{2^n n!}\left(\frac{s-1}{s+1}\right)^n [H_n(y)]^2 \exp\left(-\frac{2s}{1+s}\alpha^2\right). \tag{38}$$

This is the same as Eq. (6) of Schleich and Wheeler (1987b). We may use the probability distributions P_n in order to investigate the mean value of the various quantum functions such as quantum number n and quantum energies E_n

$$\bar{n} = \sum_n n P_n, \tag{39}$$

$$\bar{E} = \sum_n E_n P_n. \tag{40}$$

The fluctuation of $\bar{E}$ is

$$\Delta E = \left\{\sum_n E_n^2 P_n - \left(\sum_n E_n P_n\right)^2\right\}^{1/2}. \tag{41}$$

5. SEMICLASSICAL DISTRIBUTION FUNCTION: INTERFERENCE IN PHASE SPACE

Schleich and Wheeler have shown that the semiclassical approach of the interference in the squeezed state of standard harmonic oscillator gives a nice representation of quantum distribution functions in the form (Schleich *et al.*, 1988; Schleich and Wheeler, 1987a,b)

$$P_n = \left|\sqrt{\mathsf{A}_n} e^{i\theta_n} + \sqrt{\mathsf{A}_n} e^{-i\theta_n}\right|^2, \tag{42}$$

where A_n and θ_n are areas of overlap and phases associated with each overlap. This approach may give insight into the concepts of interference in phase space and into the behavior of energy distribution. We showed that this idea can also be extended to that of general TDHS in Appendix A. Now, we investigate A_n and θ_n

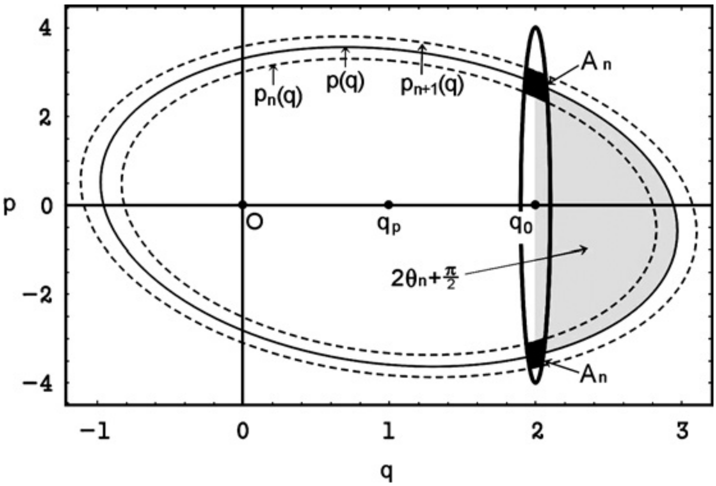

Fig. 1. Plot of the phase space for the Caldirola–Kanai oscillator which is a good example of the TDHS. We used $n = 3$, $m = 1$, $\omega_0 = 1$, $\omega_1 = 0.1$, $\gamma = 0.3$, $\hbar = 1$, $F_0 = 1$, $\vartheta = 0$, and $t = 2$.

for the general TDHS. From Eqs. (2) and (9), we derive the momentum as

$$p(q) = \frac{1}{\rho}\left[\hbar\Omega\left(n+\frac{1}{2}\right) - \frac{\Omega^2}{4\rho^2}(q-q_p)^2\right]^{1/2} - \frac{1}{2A\rho}(2B\rho - \dot{\rho})(q-q_p) + p_p. \tag{43}$$

We show $p(q)$ in Fig. 1 as a solid central line for the Caldirola–Kanai oscillator that is presented in Appendix B.

In Fig. 1, inner and outer lines correspond to

$$p_n(q) = \frac{1}{\rho}\left[\hbar\Omega n - \frac{\Omega^2}{4\rho^2}(q-q_p)^2\right]^{1/2} - \frac{1}{2A\rho}(2B\rho - \dot{\rho})(q-q_p) + p_p, \tag{44}$$

$$p_{n+1}(q) = \frac{1}{\rho}\left[\hbar\Omega(n+1) - \frac{\Omega^2}{4\rho^2}(q-q_p)^2\right]^{1/2} - \frac{1}{2A\rho}(2B\rho - \dot{\rho})(q-q_p) + p_p. \tag{45}$$

Then the weighted overlap in Fig. 1 is (Krähmer *et al*., 1994)

$$\begin{aligned} \mathsf{A}_n &= \int_{-\infty}^{\infty} dq \int_{p_n(q)}^{p_{n+1}(q)} dp\, P_s(q,p,t) \\ &= \int_{-\infty}^{\infty} |\langle q|\beta\rangle|^2 dq \int_{p_n(q)}^{p_{n+1}(q)} |\langle p|\beta\rangle|^2\, dp. \end{aligned} \tag{46}$$

When we recall Eq. (25), the integration over q may be performed by replacing the position distribution with a δ-function at $q = q_0$, where

$$q_0 = q_p + 2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha, \tag{47}$$

in the limit of $s \gg 1$. Then, the rest integral is

$$\mathsf{A}_n = \int_{p_n(q_0)}^{p_{n+1}(q_0)} |\langle p|\beta\rangle|^2 dp. \tag{48}$$

For $s \gg 1$, the exponential factor in Eq. (29) is slowly varying between the integration intervals

$$\delta p_n(q_0) = [p_{n+1}(q) - p_n(q)]_{q=q_0}. \tag{49}$$

Hence, we approximate Eq. (48) as

$$\mathsf{A}_n \simeq [|\langle p|\beta\rangle|^2]_{p=\bar{p}_n(q_0)} \times \delta p_n(q_0), \tag{50}$$

where $\bar{p}_n(q_0)$ are the mean values given by

$$\bar{p}_n(q_0) = \frac{1}{2}[p_n(q) + p_{n+1}(q)]_{q=q_0}. \tag{51}$$

By taking advantage of Eqs. (44), (45), and (47), Eqs. (49) and (51) can be rewritten as

$$\delta p_n(q_0) = \frac{\sqrt{\hbar\Omega}}{\rho}[(n+1-\mathrm{Re}\,\alpha)^{1/2} - (n-\mathrm{Re}\,\alpha)^{1/2}], \tag{52}$$

$$\bar{p}_n(q_0) = \frac{\sqrt{\hbar\Omega}}{2\rho}[(n-\mathrm{Re}\,\alpha)^{1/2} + (n+1-\mathrm{Re}\,\alpha)^{1/2}] - \frac{2B\rho - \dot{\rho}}{A}\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha + p_p. \tag{53}$$

By substitution of the above two equations into Eq. (50) we obtain

$$\mathsf{A}_n = \sqrt{\frac{2}{\pi}}\left(s + \frac{\rho^2}{sA^2\Omega^2}(2B\rho - \dot{\rho})^2\right)^{-1/2}[(n-\mathrm{Re}\,\alpha)^{1/2} + (n+1-\mathrm{Re}\,\alpha)^{1/2}]^{-1} \times \exp\left\{-\frac{1}{2}\left(s + \frac{\rho^2(2B\rho-\dot{\rho})^2}{sA^2\Omega^2}\right)^{-1} \times\{[(n-\mathrm{Re}\,\alpha)^{1/2} + (n+1-\mathrm{Re}\,\alpha)^{1/2}] - 2\,\mathrm{Im}\,\alpha\}^2\right\}. \tag{54}$$

Note that this is the same as Eq. (70) in Appendix A.

Now we calculate the phases θ_n associated with the shaded area in Fig. 1 (Krähmer *et al.*, 1994).

$$\theta_n = \frac{1}{\hbar}\int_{q_0}^{q_1} p_n(q)\,dq - \frac{\pi}{4}. \tag{55}$$

Since momentum banishes ($p = 0$) at the turning point of oscillator we obtain second order equation with respect to $q - q_p$ from Eqs. (2) and (9):

$$\left(\frac{\Omega^2}{4\rho^2} + \frac{1}{4A^2}(2B\rho - \dot{\rho})^2\right)(q - q_p)^2 - \frac{\rho p_p}{A}(2B\rho - \dot{\rho})(q - q_p) + \rho^2 p_p^2 - \hbar\Omega\left(n + \frac{1}{2}\right) = 0. \tag{56}$$

By solving the above equation we derive the upper bound of coordinate in the integration regime of Eq. (55) as

$$q_1 = \left(\frac{\Omega^2}{2\rho^2} + \frac{1}{2A^2}(2B\rho - \dot{\rho})^2\right)^{-1}\left\{\frac{\rho p_p}{A}(2B\rho - \dot{\rho}) + \Omega\left[\left(\frac{\hbar\Omega}{\rho^2} + \frac{\hbar}{A^2\Omega}(2B\rho - \dot{\rho})^2\right)\left(n + \frac{1}{2}\right) - p_p^2\right]^{1/2}\right\} + q_p. \tag{57}$$

Then the performance of integration in Eq. (55) gives

$$\theta_n = \frac{\Omega}{4\rho^2\hbar}\left\{(q_1 - q_p)\sqrt{\frac{4n\hbar\rho^2}{\Omega} - (q_1 - q_p)} - (q_0 - q_p)\sqrt{\frac{4n\hbar\rho^2}{\Omega} - (q_0 - q_p)} + \frac{4n\hbar\rho^2}{\Omega}\left[\sin^{-1}\left((q_1 - q_p)\Big/\sqrt{\frac{4n\hbar\rho^2}{\Omega}}\right) - \sin^{-1}\left((q_0 - q_p)\Big/\sqrt{\frac{4n\hbar\rho^2}{\Omega}}\right)\right]\right\} - \frac{2B\rho - \dot{\rho}}{4A\hbar\rho^2}[(q_1 - q_p)^2 - (q_0 - q_p)^2] + \frac{p_p}{\hbar}(q_1 - q_0) - \frac{\pi}{4}. \tag{58}$$

Thus, we derived semiclassical distribution functions, Eq. (42), by finding A_n and θ_n in Eqs. (54) and (58).

6. SUMMARY

In this paper, we derived the quantum probability distributions P_n of the squeezed state for the general TDHS by taking advantage of the LR invariant operator. The introduction of the LR invariant operator relieves the mathematical difficulties to evaluate quantum states of the system. We evaluated P_n in two ways in the squeezed state. Equation (35) is obtained from the exact quantum theory. On the other hand, Eq. (42) with Eqs. (54) and (58) is derived from semiclassical approach of the interference. The semiclassical approach gives deep insight into the concept of interference in phase space and into the behavior of energy distribution. We can use P_n in order to investigate the mean value of the various quantum functions such as quantum number n and quantum energies E_n. The squeezed state which is the eigenstate of the operator $\hat{b}$ defined in Eq. (18) is derived in both q and p space. Since the absolute square of the squeezed state is Gaussian we easily identified the variation of q and p as Eqs. (26) and (30). The uncertainty relation in squeezed state given by Eq. (31) is always larger than $\hbar/2$ so that the uncertainty principle hold.

The area of overlap in the phase space may be determined by the scalar product between Fock state $|\psi_n\rangle$ and squeezed state $|\beta\rangle$. The idea (Schleich *et al.*, 1988; Schleich and Wheeler, 1987a,b) of Schleich and Wheeler that the semiclassical approach of the interference in phase space of harmonic oscillator squeezed state is expressed as Eq. (42) gives a nice representation of quantum distribution functions. We showed that this idea can be extended to general TDHS in Appendix A. The weighted overlaps A_n and phases θ_n are evaluated in Eqs. (54) and (58) in the semiclassical limit, which are main results of this paper. From Fig. 2, we see that the probability that the highly squeezed TDHS is in its nth energy eigenstates oscillate. This agrees well with the reports (Schleich *et al.*,

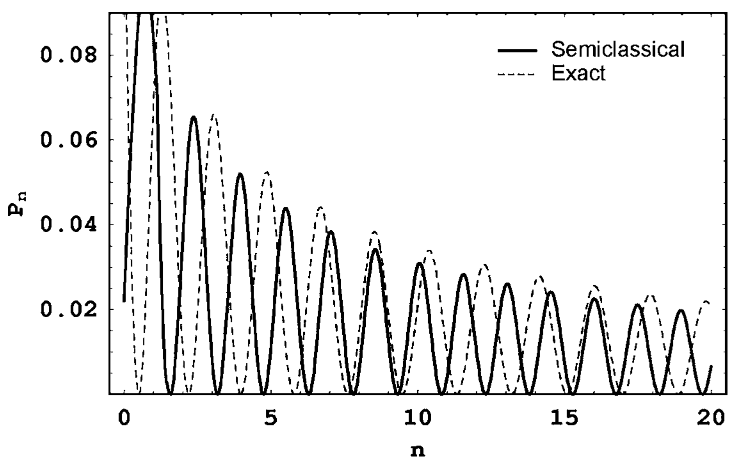

Fig. 2. Quantum distribution function P_n in a highly squeezed state of Caldirola–Kanai oscillator for $\alpha = e^{-i\omega_d t}$, $s = 150$, $\gamma = 0.4$, $\hbar = 1$, $\Omega = 1$, $m = 1$, $F_0 = 1$, $\omega_0 = 1$, $\omega_1 = 0.5$, $\vartheta = 0$, and $t = 2$. We used Eq. (42) with Eqs. (99) and (100) for solid line, and used Eq. (35) for dashed line.

1988; Schleich and Wheeler, 1987a,b) of Schleich and Wheeler for the simple harmonic oscillator.

APPENDIX A

In this appendix, we show that the concept of Eq. (42) can be extended to the general TDHS using semiclassical approach. The Wigner distribution of TDHS is given by (Choi, 2005)

$$W_n(q,p,t) = \frac{(-1)^n}{\pi\hbar} e^{-2I(q,p,t)/(\hbar\Omega)} L_n\left(\frac{4}{\hbar\Omega} I(q,p,t)\right), \tag{59}$$

where $I(q,p,t)$ is a classical invariant quantity:

$$\begin{aligned} I(q,p,t) = {} & \frac{\Omega^2}{4\rho^2(t)}(q-q_p(t))^2 \\ & + \left[\rho(p-p_p(t)) + \frac{1}{2A}(2B\rho(t)-\dot{\rho}(t))(q-q_p(t))\right]^2. \end{aligned} \tag{60}$$

The outermost wave front of Wigner distribution functions W_n are always crest and the followed inner waves form troughs and crests alternatively. The probabilities P_n are given by the overlap in phase space between W_n and distribution P_s in Eq. (32). The product $W_n P_s$ forms not only two diamond-shaped patches by the overlap between outermost wave crest and P_s, but also create ditches and tongues by the overlap between inner waves and P_s (Schleich *et al.*, 1988). Therefore, the probabilities P_n consist of two parts, i.e., "diamond" probabilities P_n^{diam} and "ditch" probabilities P_n^{ditch}

$$P_n^{\text{diam}} = 2\pi\hbar \int_{-\infty}^{\infty} dq \int_{\bar{p}_n}^{\infty} dp\, W_n(q,p,t) P_s(q,p,t), \tag{61}$$

$$P_n^{\text{ditch}} = 2\pi\hbar \int_{-\infty}^{\infty} dq \int_{-\bar{p}_n}^{\bar{p}_n} dp W_n(q,p,t) P_s(q,p,t). \tag{62}$$

First, we evaluate the diamond probabilities. The substitution of Eqs. (32) and (59) into Eq. (61) leads to

$$\begin{aligned} P_n^{\text{diam}} = {} & \frac{2(-1)^n\Omega}{\hbar\pi\rho}\left(\frac{\Omega^2}{\rho^2} + \frac{(2B\rho-\dot{\rho})^2}{s^2A^2}\right)^{-1/2} \\ & \times \int_{-\infty}^{\infty} dq \int_{\bar{p}_n}^{\infty} dp e^{-2I(q,p,t)/(\hbar\Omega)} L_n\left(\frac{4}{\hbar\Omega} I(q,p,t)\right) \\ & \times \exp\left[-\frac{\Omega s}{2\rho^2\hbar}\left(q - q_p - 2\rho\sqrt{\frac{\hbar}{\Omega}}\text{Re}\,\alpha\right)^2\right] \end{aligned}$$

$$\times \exp\left\{-\frac{2\Omega}{\hbar}\left(\frac{s\Omega^2}{\rho^2}+\frac{(2B\rho-\dot{\rho})^2}{sA^2}\right)^{-1}\right.$$
$$\left.\times\left[p-p_p-\frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha-\frac{\rho}{A\Omega}(2B\rho-\dot{\rho})\mathrm{Re}\,\alpha\right)\right]^2\right\}. \quad (63)$$

We approximate the two peaks in the diamond-shaped region as a δ-function in the coordinate space:

$$\exp\left[-\frac{\Omega s}{2\rho^2\hbar}\left(q-q_p-2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)^2\right]$$
$$\simeq\left(\frac{2\rho^2\hbar\pi}{\Omega s}\right)^{1/2}\delta\left(q-q_p-2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right). \quad (64)$$

Then, we can easily integrate Eq. (63) with respect to q. Now we introduce the following variable:

$$\eta = 4\,\mathrm{Re}\,\alpha+\frac{4}{\hbar\Omega}\left[\rho(p-p_p(t))+\frac{\rho}{A}(2B\rho(t)-\dot{\rho}(t))\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha\right]^2, \quad (65)$$

in order to integrate with respect to p, and thus

$$P_n^{\mathrm{diam}}=\frac{(-1)^n\Omega}{\rho\sqrt{2s\pi}}\left(\frac{\Omega^2}{\rho^2}+\frac{(2B\rho-\dot{\rho})^2}{s^2A^2}\right)^{-1/2}$$
$$\times\exp\left\{-\frac{2\Omega}{\hbar}\left(\frac{s\Omega^2}{\rho^2}+\frac{(2B\rho-\dot{\rho})^2}{sA^2}\right)^{-1}\right.$$
$$\left.\times\left[\frac{\sqrt{\hbar\Omega}}{2\rho}[(n-\mathrm{Re}\,\alpha)^{1/2}+(n+1-\mathrm{Re}\,\alpha)^{1/2}]-\frac{1}{\rho}\sqrt{\hbar\Omega}\,\mathrm{Im}\,\alpha\right]^2\right\}$$
$$\times\int_{\bar{\eta}_n}^{\infty}d\eta\frac{1}{\sqrt{\eta-4\mathrm{Re}\,\alpha}}e^{-\eta/2}L_n(\eta), \quad (66)$$

where

$$\bar{\eta}_n=4\,\mathrm{Re}\,\alpha+\frac{4}{\hbar\Omega}\left[\rho(\bar{p}_n-p_p(t))+\frac{\rho}{A}(2B\rho(t)-\dot{\rho}(t))\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha\right]^2$$
$$=4\,\mathrm{Re}\,\alpha+[(n-\mathrm{Re}\,\alpha)^{1/2}+(n+1-\mathrm{Re}\,\alpha)^{1/2}]^2. \quad (67)$$

Since $\bar{\eta}_n - 4\,\mathrm{Re}\,\alpha$ varies slowly at the turning point $\eta = \bar{\eta}_n$ the square root of it in Eq. (66) may be out of the integral with no wonder

$$\int_{\bar{\eta}_n}^{\infty} d\eta \frac{1}{\sqrt{\eta - 4\mathrm{Re}\,\alpha}} e^{-\eta/2} L_n(\eta) \simeq \frac{1}{\sqrt{\bar{\eta}_n - 4\mathrm{Re}\,\alpha}} \int_{\bar{\eta}_n}^{\infty} d\eta e^{-\eta/2} L_n(\eta). \tag{68}$$

The asymptotic solution of the remnant integral is known in the limit $n \to \infty$ (Schleich *et al.*, 1988)

$$\int_{\bar{\eta}_n}^{\infty} d\eta\, e^{-\eta/2} L_n(\eta) \simeq 2(-1)^n. \tag{69}$$

Thus, we arrive at

$$\begin{aligned} P_n^{\mathrm{diam}} &= \sqrt{\frac{2}{\pi}} \left(s + \frac{\rho^2}{sA^2\Omega^2} (2B\rho - \dot{\rho})^2 \right)^{-1/2} \\ &\quad \times [(n - \mathrm{Re}\,\alpha)^{1/2} + (n + 1 - \mathrm{Re}\,\alpha)^{1/2}]^{-1} \\ &\quad \times \exp\left\{ -\frac{1}{2} \left(s + \frac{\rho^2(2B\rho - \dot{\rho})^2}{sA^2\Omega^2} \right)^{-1} \right. \\ &\quad \left. \times \left\{ [(n - \mathrm{Re}\,\alpha)^{1/2} + (n + 1 - \mathrm{Re}\,\alpha)^{1/2}] - 2\,\mathrm{Im}\,\alpha \right\}^2 \right\} \equiv \mathsf{A}_n. \end{aligned} \tag{70}$$

This is the same as the weighted area of each zone of the two diamond-shaped patches.

Now, we calculate the ditch integral. The ditch probabilities can be obtained by subtracting the probabilities related to two diamond-shaped zones from total probabilities associated with nth number states.

$$P_n^{\mathrm{ditch}} = 2\pi\hbar \int_{-\infty}^{\infty} dq \left(\int_{-\infty}^{\infty} dp - 2 \int_{\bar{p}_n}^{\infty} dp \right) W_n(q, p, t) P_s(q, p, t). \tag{71}$$

The total probabilities may be represented as

$$\begin{aligned} P_n &= 2\pi\hbar \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W_n(q, p, t) P_s(q, p, t) \\ &= 2\pi\hbar \frac{\Omega}{2\pi\rho} \left(\frac{\Omega^2}{\rho^2} + \frac{(2B\rho - \dot{\rho})^2}{s^2A^2} \right)^{-1/2} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, W_n(q, p, t) \\ &\quad \times \exp\left[-\frac{\Omega s}{2\rho^2\hbar} \left(q - q_p - 2\rho\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha \right)^2 \right] \\ &\quad \times \exp[-A_1(p - p_p - A_2)^2], \end{aligned} \tag{72}$$

where

$$A_1 = \frac{2\Omega}{\hbar}\left(\frac{s\Omega^2}{\rho^2} + \frac{(2B\rho - \dot{\rho})^2}{sA^2}\right)^{-1}, \tag{73}$$

$$A_2 = \frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha - \frac{\rho}{A\Omega}(2B\rho - \dot{\rho})\,\mathrm{Re}\,\alpha\right). \tag{74}$$

To integrate over p we neglect the little decrease of $P_s(q, p, t)$ with increasing p along constant q

$$\exp[-A_1(p - p_p - A_2)^2] \simeq 1. \tag{75}$$

Then, the rest integral over p is just (Choi, 2005; Schleich *et al.*, 1988; Zurek, 1991)

$$\int_{-\infty}^{\infty} dp\, W_n(q, p, t) = |\langle q|\psi_n(t)\rangle|^2. \tag{76}$$

Here, $\langle q|\psi_n(t)\rangle$ are the coordinate wave functions given by Eq. (11). However, in order to investigate phase properties of the system, it is very useful to replace exact $\langle q|\psi_n(t)\rangle$ with the corresponding WKB wave functions that given by (Choi, 2004c; Schleich *et al.*, 1988)

$$\langle q|\psi_n(t)\rangle \simeq \sqrt{\frac{2\Omega}{\pi\hbar\rho^2}}\frac{1}{(\bar{p}_n')^{1/2}}\ \cos\,\theta_n(q), \tag{77}$$

where

$$\bar{p}_n' = \bar{p}_n - p_p - A_2, \tag{78}$$

$$\theta_n(q) = \frac{1}{\hbar}\int_{q_0}^{q_1} p_n(q)\,dq - \frac{\pi}{4}. \tag{79}$$

Then, a little algebra after performing integration over p leads to

$$P_n = \frac{\Omega}{\pi\rho^2}\left(1 + \frac{\rho^2(2B\rho - \dot{\rho})^2}{s^2A^2\Omega^2}\right)^{-1/2}\frac{1}{\bar{p}_n'}\int_{-\infty}^{\infty} dq\{1 + \cos[2\theta_n(q)]\}$$
$$\times\ \exp\left[-\frac{\Omega s}{2\rho^2\hbar}\left(q - q_p - 2\rho\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha\right)^2\right] \simeq 2P_n^{\mathrm{diam}} + P_n^{\mathrm{ditch}}. \tag{80}$$

Note that the term containing second integral is ditch probability density (Schleich *et al.*, 1988) so that

$$P_n^{\mathrm{ditch}} = \frac{\Omega}{\pi\rho^2}\left(1 + \frac{\rho^2(2B\rho - \dot{\rho})^2}{s^2A^2\Omega^2}\right)^{-1/2}\frac{1}{\bar{p}_n'}\int_{-\infty}^{\infty} dq\,\cos[2\theta_n(q)]$$

$$\times \ \exp\left[-\frac{\Omega s}{2\rho^2\hbar}\left(q - q_p - 2\rho\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha \right)^2 \right]. \tag{81}$$

With the help of the integral formula (Schleich *et al.*, 1988)

$$\int_{-\infty}^{\infty} dq f(q) e^{-a(q-b)^2} = \sqrt{\frac{\pi}{a}} \sum_{j=0}^{\infty} \frac{1}{j!(4a)^j} \left. \frac{d^{2j} f(q)}{dq^{2j}} \right|_{q=b}, \tag{82}$$

the integral in Eq. (81) becomes

$$\begin{aligned} I^{\text{ditch}} &\equiv \int_{-\infty}^{\infty} dq \cos[2\theta_n(q)] \exp\left[-\frac{\Omega s}{2\rho^2\hbar}\left(q - q_p - 2\rho\sqrt{\frac{\hbar}{\Omega}}\,\mathrm{Re}\,\alpha \right)^2 \right] \\ &= \sqrt{\frac{2\pi\rho^2\hbar}{\Omega s}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\rho^2\hbar}{2\Omega s} \right)^j \left. \frac{d^{2j}\cos[2\theta_n(q)]}{dq^{2j}} \right|_{q=q_p+2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha}. \end{aligned} \tag{83}$$

If we ignore the slow variation of $\bar{p}_n'(q)$, the differentiation in the above equation becomes

$$\frac{d^{2j}\cos[2\theta_n(q)]}{dq^{2j}} \simeq (-4)^j \left(\frac{\bar{p}_n'(q)}{\hbar} \right)^{2j} \cos[2\theta_n(q)], \tag{84}$$

so that

$$I^{\text{ditch}} = \sqrt{\frac{2\pi\rho^2\hbar}{\Omega s}} \exp\left[-\frac{2\rho^2\hbar}{\Omega s}\left(\frac{\bar{p}_n'(q_0)}{\hbar} \right)^2 \right] \cos[2\theta_n(q_0)]. \tag{85}$$

Making use of Eq. (85), Eq. (81) can be represented as

$$\begin{aligned} P_n^{\text{ditch}} \approx\ & 2\sqrt{\frac{2}{\pi}}\left(s + \frac{\rho^2}{sA^2\Omega^2}(2B\rho - \dot{\rho})^2 \right)^{-1/2} \\ & \times[(n - \mathrm{Re}\,\alpha)^{1/2} + (n + 1 - \mathrm{Re}\,\alpha)^{1/2}]^{-1} \\ & \times \exp\left\{ -\frac{1}{2}\left(s + \frac{\rho^2(2B\rho - \dot{\rho})^2}{sA^2\Omega^2} \right)^{-1} \right. \\ & \times \{[(n - \mathrm{Re}\,\alpha)^{1/2} + (n + 1 - \mathrm{Re}\,\alpha)^{1/2}] \\ & \left. - 2\,\mathrm{Im}\,\alpha\}^2 \right\} \cos[2\theta_n(q_0)] = 2\mathsf{A}_n \cos[2\theta_n(q_0)]. \end{aligned} \tag{86}$$

Recalling second relation in Eq. (80) with Eqs. (70) and (86), we have

$$\begin{aligned} P_n &= 2\mathsf{A}_n[1 + \cos(2\theta_n)] \\ &= \left| \sqrt{\mathsf{A}_n} e^{i\theta_n} + \sqrt{\mathsf{A}_n} e^{-i\theta_n} \right|^2. \end{aligned}$$

Thus, we obtained the probabilities in Eq. (42).

APPENDIX B

In this appendix, we apply our development to the Caldirola–Kanai oscillator (Kanai, 1948). With the choice of

$$A(t) = e^{-\gamma t}\frac{1}{2m}, \tag{87}$$

$$C(t) = e^{\gamma t}\frac{1}{2}m\omega_0^2, \tag{88}$$

$$D(t) = -e^{\gamma t}F_0\cos(\omega_1 t + \vartheta), \tag{89}$$

where m is the mass, γ is the damping constant, ω_0 and ω_1 are some constant frequencies, and F_0 and ϑ are the amplitude and the initial phase of the driving force, and $B(t)$, $E(t)$, and $F(t)$ are zero, Eq. (1) becomes

$$\hat{H} = e^{-\gamma t}\frac{\hat{p}^2}{2m} + e^{\gamma t}\frac{1}{2}m\omega_0^2\hat{q}^2 - e^{\gamma t}F_0\cos(\omega_1 t + \vartheta)\hat{q}. \tag{90}$$

Then, the solutions of Eqs. (3)–(5) are (Choi, 2004a; Pedrosa *et al.*, 1997)

$$\rho(t) = \sqrt{\frac{\Omega}{2m\omega_d}e^{-\gamma t/2}}, \tag{91}$$

$$q_p(t) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega_1^2\right)^2 + \gamma^2\omega_1^2}}\cos(\omega_1 t + \vartheta - \delta), \tag{92}$$

$$p_p(t) = -\frac{F_0\omega_1}{\sqrt{\left(\omega_0^2 - \omega_1^2\right)^2 + \gamma^2\omega_1^2}}e^{\gamma t}\sin(\omega_1 t + \vartheta - \delta), \tag{93}$$

where modified frequency ω_d and phase δ are given by

$$\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}, \tag{94}$$

$$\delta = \tan^{-1}\frac{\gamma\omega_1}{\omega_0^2 - \omega_1^2}. \tag{95}$$

In this case Eqs. (43)–(45) become

$$p(q) = \sqrt{2m\omega_d}e^{\gamma t/2}\left[\hbar\left(n + \frac{1}{2}\right) - \frac{m\omega_d}{2}e^{\gamma t}(q - q_p)^2\right]^{1/2} - \frac{\gamma m}{2}e^{\gamma t}(q - q_p) + p_p, \tag{96}$$

$$p_n(q) = \sqrt{2m\omega_d} e^{\gamma t/2} \left[\hbar n - \frac{m\omega_d}{2} e^{\gamma t} (q - q_p)^2 \right]^{1/2} - \frac{\gamma m}{2} e^{\gamma t} (q - q_p) + p_p, \tag{97}$$

$$p_{n+1}(q) = \sqrt{2m\omega_d} e^{\gamma t/2} \left[\hbar (n+1) - \frac{m\omega_d}{2} e^{\gamma t} (q - q_p)^2 \right]^{1/2} - \frac{\gamma m}{2} e^{\gamma t} (q - q_p) + p_p. \tag{98}$$

Eventually, for the Caldirola–Kanai oscillator, Eqs. (54) and (58) reduce to

$$\begin{aligned} \mathsf{A}_n = & \sqrt{\frac{2}{\pi}} \left(s + \frac{\gamma^2}{4\omega_d^2 s} \right)^{-1/2} [(n - \operatorname{Re}\alpha)^{1/2} + (n + 1 - \operatorname{Re}\alpha)^{1/2}]^{-1} \\ & \times \exp \left\{ -\frac{1}{2} \left(s + \frac{\gamma^2}{4\omega_d^2 s} \right)^{-1} \{ [(n - \operatorname{Re}\alpha)^{1/2} \right. \\ & \left. + (n + 1 - \operatorname{Re}\alpha)^{1/2}] - 2 \operatorname{Im}\alpha \}^2 \right\}, \end{aligned} \tag{99}$$

$$\begin{aligned} \theta_n = & \frac{m\omega_d}{2\hbar} e^{\gamma t} \left\{ (q_1 - q_p) \sqrt{\frac{2n\hbar}{m\omega_d e^{\gamma t}} - (q_1 - q_p)} \right. \\ & - (q_0 - q_p) \sqrt{\frac{2n\hbar}{m\omega_d e^{\gamma t}} - (q_0 - q_p)} \\ & + \frac{2n\hbar}{m\omega_d} e^{-\gamma t} \left[\sin^{-1} \left((q_1 - q_p) \sqrt{\frac{m\omega_d}{2n\hbar} e^{\gamma t}} \right) \right. \\ & \left. \left. - \sin^{-1} \left((q_0 - q_p) \sqrt{\frac{m\omega_d}{2n\,\hbar} e^{\gamma t}} \right) \right] \right\} - \frac{m\gamma}{4\,\hbar} \sqrt{\frac{2m\omega_d}{\Omega}} e^{3\gamma t/2} [(q_1 - q_p)^2 \\ & - (q_0 - q_p)^2] + \frac{p_p}{\hbar} (q_1 - q_0) - \frac{\pi}{4}, \end{aligned} \tag{100}$$

with

$$q_0 = \sqrt{\frac{2\hbar}{m\omega_d}} e^{-\gamma t/2} \operatorname{Re}\alpha + q_p, \tag{101}$$

$$q_1 = \frac{\omega_d e^{-\gamma t}}{m\omega_0^2} \left\{ \frac{\gamma}{2\omega_d} p_p + \left[\frac{2m\hbar}{\omega_d} e^{\gamma t} \omega_0^2 \left(n + \frac{1}{2} \right) - p_p^2 \right]^{1/2} \right\} + q_p. \tag{102}$$

We represented P_n for the Caldirola–Kanai oscillator by use of Eqs. (99) and (100) in Fig. 2.

REFERENCES

Bandyopadhyay, J. N., Lakshminarayan, A., and Sheorey, V. B. (2001). Algebraic approach in the study of time-dependent nonlinear integrable systems: Case of the singular oscillator. *Physical Review A* **63**, 042109.

Chaturvedi, S., Milburn, G. J., and Zhang, Z. (1998). Interference in hyperbolic space. *Physical Review A* **57**, 1529–1535.

Choi, J. R. (2004c). WKB wave function of the general time-dependent quadratic Hamiltonian system. *International Journal of Theoretical Physics* **43**, 947–958.

Choi, J. R. (2004a). Coherent states of general time-dependent harmonic oscillator. *Pramana—Journal of Physics* **62**, 13–29.

Choi, J. R. (2004b). The dependency on the squeezing parameter for the uncertainty relation in the squeezed states of the time-dependent oscillator. *International Journal of Modern Physics B* **18**, 2307–2324.

Choi, J. R. (2005). Wigner distribution function for the time-dependant quadratic-Hamiltonian quantum system using the Lewis-Riesenfeld invariant operator. *International Journal of Theoretical Physics*. **44**, 327–348.

Dodonov, V. V. and Man'ko, V. I. (1978). Loss energy states of nonstationary quantum systems. *Il Nuovo Cimento* **44**, 265–273.

El-Orany, F. A. A., Mahran, M. H., Wahiddin, M. R. B., and Hashim, A. M. (2004). Quantum phase properties of two-mode Jaynes–Cummings model for Schrödinger-cat states: Interference and entanglement. *Optics Communications* **240**, 169–184.

Kanai, E. (1948). On the quantization of dissipative systems. *Progress of Theoretical Physics* **3**, 440–442.

Kim, S. P. and Lee, C. H. (2000). Nonequilibrium quantum dynamics of second order phase space transition. *Physical Review A* **62**, 125020.

Krähmer, D., Mayr, E., Vogel, K., and Schleich, W. P. (1994). Meet a squeezed state and interfere in phase space. In *Current Trends in Optics*, edited by Dainty, J. C. Academic Press, London, Chap. 3, pp. 37–50.

Lassig, C. C. and Milburn, G. J. (1993). Interference in a spherical phase space and asymptotic behavior of the rotation matrices. *Physical Review A* **48**, 1854–1860.

Lewis, H. R., Jr. and Riesenfeld, W. B. (1969). An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field. *Journal of Mathematical Physics* **10**, 1458–1473.

Nieto, M. M. and Truax, D. R. (2000). Time-dependent Schrodinger equations with anisotropic potentials and magnetic fields. *Journal of Mathematical Physics* **41**, 2741–2752.

Pedrosa, I. A. and Guedes, I. (2002). Wave function of the time-dependent inverted harmonic oscillator. *Modern Physics Letters B* **16**, 637–643.

Pedrosa, I. A., Serra, G. P., and Guedes, I. (1997). Wave functions of a time-dependent harmonic oscillator with and without a singular perturbation. *Physical Review A* **56**, 4300–4303.

Šamaj, L. (2002). Evolution of quantum systems with a scaling type time-dependent Hamiltonians. *International Journal of Modern Physics B* **16**, 3909–3914.

Schleich, W., Walls, D. F., and Wheeler, J. A. (1988). Area of overlap and interference in phase space versus Wigner pseudoprobabilities. *Physical Review A* **38**, 1177–1186.

Schleich, W. and Wheeler, J. A. (1987a). Oscillations in photon distribution of squeezed states and interference in phase space. *Nature* **326**, 574–577.

Schleich, W. and Wheeler, J. A. (1987b). Oscillations in photon distribution of squeezed states. *Journal of the Optical Society of America B* **4**, 1715–1722.

Song, D. Y. (2000). Periodic Hamiltonian and Berry's phase in harmonic oscillators. *Physical Review A* **61**, 024102.

Um, C. I., Yeon, K. H., and George, T. F. (2002). The quantum damped harmonic oscillator. *Physics Reports* **362**, 63–192.

Wei, L. F., Wang, S. J., and Lei, X. L. (2002). Gauge-covariant properties of a linear nonautonomous quantum system: Time-dependent even and odd coherent states. *Journal of Physics A: Mathematical and General* **35**, 435–445.

Zurek, W. H. (1991). Decoherence and the transition from quantum to classical. *Physics Today* **October**, 36–44.